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Double-boundary-layer concept in free-convection at high Prandtl numbers

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The double-boundary-layer-concept of Stewartson & Jones (1957) for free-convection at high Prandtl numbers (σ) is extended. By the method of matched asymptotic expansions it is shown that series-solutions in powers of $\sigma^{-\frac{1}{4}}$ exist. Calculations are carried upto terms $0(\sigma^{-1})$. The Nusselt number is seen to be in an error of 3.5% compared with exact solutions of Ostrach (1953), for a Prandtl number as low as 1.

Introduction

It is well-known that in problems of free-convective heat transfer, from or to a heated surface, the Prandtl number (σ) of the fluid medium plays a very important role. Mathematically, this role is expressed through the occurrence of this number in the governing non-dimensional partial differential equations. Solutions of these equations have been obtained by different methods. One class of solutions, obtained by integrating the partial differential equations by the integral method of Kármán-Pohlhausen, has an advantage in that they exhibit the Prandtl number dependence explicitly; but at the same time they provide with inaccurate results for skin-friction and heat transfer. Reference may be made to Eckert & Gross (1963) for examples of this type. On the other hand, Ostrach (1953), and Sparrow & Gregg (1958), and others have reduced the partial differential equations through similarity transformations to ordinary differential equations which they solved on high-speed computers. A disadvantage of this technique is that the results have to be tabulated for distinct values of the Prandtl number A third technique maintaining the advantages of both the methods, *viz.* the accuracy and the explicit Prandtl number dependence has been developed by Le Fevre (1956), and Stewartson & Jones (1957). In fact, Stewartson & Jones in a study of free-convective heat transfer from a vertical flat plate obtained solutions for the extreme case $\sigma = \infty$. They introduced a concept of two boundary layers, one of thickness $0(\sigma^{-1/4})$ in which the temperature difference was brought to zero, and one of thickness $0(\sigma^{1/4})$ in which the component of the velocity parallel to the surface was brought to zero again.

We shall in this paper extend their work and obtain series-solutions in powers of $\sigma^{-\frac{1}{4}}$, to terms $0(\sigma^{-1})$, by means of the method of matched asymptotic expansions (Van Dyke, 1964). We shall call the layers the inner and outer layers respectively.

Mathematical formulation

Free-convection from or to a uniformly heated semi-infinite vertical flat surface is governed by the following boundary layer equations

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}\pm g\rho\beta\theta,\\
&u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\alpha\frac{\partial^2 \theta}{\partial y^2}\\
&u=v=0,\quad \theta=t_w-t_\infty,\quad y=0,\\
&u\to 0,\quad \theta\to 0\quad \text{as}\quad y\to\infty,
\end{aligned}\right\} \qquad \dots \quad (1)$$

where u, v are the velocity components, θ is the temperature difference $(t-t_\infty)$, x is measured from the leading edge along the plate and y is the distance out perpendicular to the plate, the plus and minus signs apply for heating and cooling of the fluid respectively, and the other symbols have their usual meanings

A similarity solution of (1) is given by

$$\left.\begin{aligned}
&u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x},\\
&\phi=\frac{t-t_\infty}{t_w-t_\infty},\quad \psi=2\sqrt{2}\nu(Gr_x)^{1/4}f,\\
&Gr_x=|g\beta x^3(t_w-t_\infty)/\nu^2|,\\
&\eta=\frac{y}{x}\left(\frac{Gr_x}{4}\right)^{1/4},
\end{aligned}\right\} \qquad \dots \quad (2)$$

where f and ϕ are functions of η alone satisfying the equations

$$\frac{d^3f}{d\eta^3}+3f\frac{d^2f}{d\eta^2}-2\left(\frac{df}{d\eta}\right)^2+\phi=0, \qquad \dots \quad (3)$$

and

$$\frac{d^2\phi}{d\eta^2}+3\sigma f\frac{d\phi}{d\eta}=0. \qquad \dots \quad (4)$$

The boundary conditions are

$$\left.\begin{aligned}
&f=\frac{df}{d\eta}=0,\quad \phi=1\quad \text{at}\quad \eta=0,\\
&\frac{df}{d\eta}\to 0,\quad \phi\to 0\quad \text{as}\quad \eta\to\infty.
\end{aligned}\right\} \qquad \dots \quad (5)$$

INNER AND OUTER LAYER TRANSFORMATIONS

Following Stewartson & Jones (1957), the appropriate transformations are assumed as follows .

$$\begin{aligned} \zeta_1 &= (3\sigma)^{1/4}\eta, \\ f &= (3\sigma)^{-3/4}F_1(\zeta_1), \\ \phi &= \Phi_1(\zeta_1) \end{aligned} \qquad (6)$$

Substituting in (3) and (4) we get

$$F_1''' + \Phi_1 + (3\sigma)^{-1}\{3F_1F_1'' - 2(F'_1)^2\} = 0. \qquad (7)$$

and

$$\Phi_1'' + F_1\Phi_1' = 0 \qquad (8)$$

Beyond the inner layer, ϕ becomes exponentially small and $\frac{df}{d\eta} = 0(\sigma^{-\frac{1}{2}})$. The outer layer is then introduced to reduce $\frac{df}{d\eta}$ to zero. In this layer the appropriate transformations are

$$\left.\begin{aligned} \zeta_2 &= \gamma(3\sigma)^{-1/4}\eta, \\ f &= \gamma(3\sigma)^{-1/4}F_2(\zeta_2), \\ \phi &= 0, \end{aligned}\right\} \quad \dots \quad (9)$$

where γ is an arbitrary constant to be specified later. The equation for F_2 is

$$F_2''' + 3F_2F_2'' - 2(F'_2)^2 = 0 \qquad \dots \quad (10)$$

In the above a prime means differentiation with respect to the appropriate independent variable ζ_1 or ζ_2. It should also be remarked here that the boundary conditions at infinity are redundant for F_1 and those on the surface for F_2.

Now, applying the method of matched asymptotic expansions we find that series-solutions in the following forms exist

$$\left.\begin{aligned} F_1 &= \sum_{i=0}^{\infty} (3\sigma)^{-i/2}F_{1i}, \\ \Phi_1 &= \sum_{i=0}^{\infty} (3\sigma)^{-i/2}\Phi_{1i}, \\ F_2 &= \sum_{i=0}^{\infty} (3\sigma)^{-i/2}F_{2i}, \end{aligned}\right\} \quad \dots \quad (11)$$

together with the boundary conditions :

$$
\left.
\begin{aligned}
&F_{10}(0) = 0, \quad F'_{10}(0) = 0, \quad F''_{10}(\infty) = 0;\\
&F_{20}(0) = 0, \quad F'_{20}(0) = 1, \quad F'_{20}(\infty) = 0;\\
&F_{11}(0) = 0, \quad F'_{11}(0) = 0, \quad F''_{11}(\infty) = \gamma^3 F''_{20}(0);\\
&F_{21}(0) = \lim_{\zeta_1\to\infty} \left\{\frac{1}{\gamma} F_{10}(\zeta_1) - \gamma\zeta_1\right\}\\
&F'_{21}(0) = \frac{1}{\gamma^2} \lim_{\zeta_1\to\infty} \{F'_{11}(\zeta_1) - \zeta_1 F''_{11}(\zeta_1)\}, \quad F'_{21}(\infty) = 0;\\
&F_{12}(0) = 0, \quad F'_{12}(0) = 0, \quad F''_{12}(\infty) = \gamma^3 \lim_{\zeta_1\to\infty} \{F''_{21}(0) + 2\gamma\zeta_1\};\\
&F_{22}(0) = \lim_{\zeta_1\to\infty} \left\{\frac{1}{\gamma} F_{11}(\zeta_1) - \gamma\zeta_1 F_{21}'(0) - \tfrac{1}{2}\gamma^2\zeta_1^2 F''_{20}(0)\right\},\\
&F_{22}'(0) = \lim_{\zeta_1\to\infty} \left\{\frac{1}{\gamma^2} F'_{12}(\zeta_1) - \gamma\zeta_1 F''_{21}(0) - \gamma^2\zeta_1^2\right\},\\
&F'_{22}(\infty) = 0;\\
&\Phi_{10}(0) = 1, \quad \Phi_{10}(\infty) = 0;\\
&\Phi_{11}(0) = 0, \quad \Phi_{11}(\infty) = 0;\\
&\Phi_{12}(0) = 0, \quad \Phi_{12}(\infty) = 0;
\end{aligned}
\right\}
\tag{12}
$$

where $\gamma^2 = \lim_{\zeta_1\to\infty} F'_{10}(\zeta_1)$ [See Appendix].

The equations for F's and Φ's are

$$F'''_{10} + \Phi_{10} = 0, \tag{13}$$

$$F'''_{11} + \Phi_{11} = 0, \tag{14}$$

$$F'''_{12} + \Phi_{12} + 3F_{10}F''_{10} - 2(F'_{10})^2 = 0, \tag{15}$$

$$\Gamma_1(\Phi_{10}) = 0, \tag{16}$$

$$\Gamma_1(\Phi_{11}) + \Phi'_{10}F_{11} = 0, \tag{17}$$

$$\Gamma_1(\Phi_{12}) + \Phi'_{10}F_{12} + F_{11}\Phi'_{11} = 0, \tag{18}$$

$$\Gamma_2(F_{20}) - 2(F'_{20})^2 = 0, \tag{19}$$

$$\Gamma_2(F_{21}) - 4F'_{20}F'_{21} + 3F''_{20}F_{21} = 0, \tag{20}$$

$$\Gamma_2(F_{22}) - 4F'_{20}F'_{22} + 3F''_{20}F_{22} + 3F_{21}F''_{21} - 2(F'_{21})^2 = 0, \tag{21}$$

where Γ_1 and Γ_2 are the differential operators,

$$\Gamma_1 \equiv \frac{d^2}{d\zeta_1^2} + F_{10}\frac{d}{d\zeta_1}$$

and

$$\Gamma_2 \equiv \frac{d^3}{d\zeta_2^3} + 3F_{20}\frac{d^2}{d\zeta_2^2}.$$

SOLUTION

Equations (13)–(21), subject to the boundary conditions (12), have been solved on the electronic computer IBM 7094II at The Imperial College, London. The values of the unknown boundary derivatives required to start numerical computations are presented below :

$$\left.\begin{aligned} &\gamma = 0.940347 \\ &F'''_{10}(0) = 1.085060,\; F'''_{11}(0) = -0.700126,\; F'''_{12}(0) = 0.861980; \\ &\Phi'_{10}(0) = -0.540229,\; \Phi'_{11}(0) = 0.245418,\; \Phi'_{12}(0) = -0.147591; \\ &F_{20}(0) = 0.0,\quad F''_{20}(0) = 1.0,\quad F'''_{20}(0) = -1.540791; \\ &F_{21}(0) = -0.632131,\; F''_{21}(0) = 1.142751,\; F'''_{21}(0) = -1.654336, \\ &F_{22}(0) = -1.145929,\; F''_{22}(0) = 2.583852,\; F'''_{22}(0) = -4.087507. \end{aligned}\right\} \quad (22)$$

SKIN-FRICTION AND HEAT-TRANSFER

Skin-friction is proprtional to $\left(\frac{d^2f}{d\eta^2}\right)_{\eta=0}$ which is given by

$$\left(\frac{d^2f}{d\eta^2}\right)_{\eta=0} = (3\sigma)^{-1/4}[1.085-0.700(3\sigma)^{-\frac{1}{2}}+0.862(3\sigma)^{-1}+\ldots] \quad (23)$$

The rate of heat-transfer is represented by the Nusselt number Nu that is given by

$$Nu = -\frac{x}{(t\omega-t_\infty)}\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$$

$$= (\sigma\, Gr_x)^{-1/4}[0.502-0.228(3\sigma)^{-\frac{1}{2}}+0.137(3\sigma)^{-1}+\ldots] \quad \ldots \quad (24)$$

The values of $Nu\,(Gr_x)^{1/4}$ calculated from equation (24) for different values of σ are given in the table along with those obtained by Ostrach (1953)

TABLE 1

Values of the $Nu\,(Gr_x)^{-1/4}$

σ	Roy	Ostrach (1953)
1	0.416	0.401
2	0.517	0.507
10	0.827	0.827
10^2	1.547	1.549
10^3	2.800	2.805

REMARKS

(1) It is clear from the table that though our analysis has been developed on the assumption of a large Prandtl number the values of the Nusselt number

compare very favourably with the exact values of Ostrach even for a Prandtl number as low as 1.

(2) The boundary conditions $F''_{10}(\infty) = 0$, $F_{20}(0) = 0$ and $F'_{20}(0) = 1$ have been used by Stewartson & Jones (1957). However, they have not given a reason for introducing them. It has been clear that only through the method of matched asymptotic expansions one can deduce these and the other conditions.

(3) The double-boundary-layer concept described here should be applicable to other boundary layer phenomena in free-convection from or to an isothermal surface at high Prandtl numbers. The author has used it in substantiating a conjecture of Gebhart (1962) on dissipation effects in free-convection (Roy 1969).

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Appendix—Matched asymptotic expansion

Equations (13)-(21) suggest that series expansions in negative powers of σ exist. Let us consider the equations (13) and (19) and assume that for $\sigma = \infty$, the solutions are F_{10} and F_{20} and that the next terms are $K(\sigma)F_{11}$ and $K(\sigma)F_{21}$ respectively. To find the form of $K(\sigma)$ we match the solutions for large values of ζ_1 and for small values of ζ_2. Thus

$$(3\sigma)^{-3/4}F_1(\zeta_1) \text{ for large } \zeta_1$$
$$\approx \gamma(3\sigma)^{-1/4}F_2(\zeta_2) \text{ for small } \zeta_2.$$

Therefore, $F_{10}(\zeta_1)+K(\sigma)F_{11}(\zeta_1)+\ldots$ for large ζ_1

$$\approx \gamma(3\sigma)^{\frac{1}{2}}\{F_{20}(\zeta_2)+K(\sigma)F_{21}(\zeta_2)+\ldots\} \text{ for small } \zeta_2 \qquad \ldots \quad (\text{A }1)$$

We now assume that for small ζ_2, F_{20} and F_{21} have Taylor-series representations,

$$\left.\begin{aligned} F_{20} &= a_0+a_1\zeta_2+a_2\zeta_2^{\,2}+\ldots \\ F_{21} &= b_0+b_1\zeta_2+b_2\zeta_2^{\,2}+\ldots \end{aligned}\right\} \qquad (\text{A}.2)$$

Putting from (A.2) into (A.1), and noting that $\zeta_2 = \gamma(3\sigma)^{-\frac{1}{2}}\zeta_1$, we get

$$F_{10}(\zeta_1)+K(\sigma)\,F_{11}(\zeta_1)+\ldots$$
$$\sim \gamma(3\sigma)^{\frac{1}{2}}[a_0+\gamma a_1(3\sigma)^{-\frac{1}{2}}\zeta_1+\gamma^2 a_2(3\sigma)^{-1}\zeta_1^{\,2}+\ldots$$
$$+K(\sigma)\{b_0+\gamma b_1(3\sigma)^{-\frac{1}{2}}\zeta_1+\gamma^2 b_2(3\sigma)^{-1}\zeta_1^{\,2}+\ldots\}] \quad \text{as} \quad \zeta_1 \to \infty \qquad (\text{A}.3)$$

(A.3) shows that

(i) $a_0 = 0$, since F_{10} cannot behave as $\sigma^{\frac{1}{3}}$ as $\zeta_1 \to \infty$, and (ii) $K(\sigma) = (3\sigma)^{-1}$. Proceeding in this way it is seen that the third term is proportional to $(3\sigma)^{-1}$, and so on. Further, from (A.3) we get,

$$\left.\begin{aligned} F_{10} &\sim \gamma b_0 + \gamma^2 a_1 \zeta_1, \\ F_{11} &\sim \gamma c_0 + \gamma^2 b_1 \zeta_1 + \gamma^3 a_2 \zeta_1^2, \\ &\qquad\qquad \text{as } \zeta_1 \to \infty \\ \text{where} \quad F_{22} &= c_0 + c_1 \zeta_2 + \ldots \qquad \text{as } \zeta_2 \to 0 \end{aligned}\right\} \tag{A.4}$$

From (A.2) and (A.4) the boundary conditions for F_{10} and F_{11} at $\zeta_1 = \infty$ and those for F_{20} and F_{21} at $\zeta_2 = 0$ follow. An extension of this matching technique leads to the rest of the boundary conditions.